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LMI-Based Neurocontroller for State-Feedback Guaranteed Cost Control of Discrete-Time Uncertain System

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SUMMARY The application of neural networks to the state-feedback guaranteed cost control problem of discrete-time system that has uncertainty in both state and input matrices is investigated. Based on the Linear Matrix Inequality (LMI) design, a class of a state feedback controller is newly established, and sufficient conditions for the existence of guaranteed cost controller are derived. The novel contribution is that the neurocontroller is substituted for the additive gain perturbations. It is newly shown that although the neurocontroller is included in the discrete-time uncertain system, the robust stability for the closed-loop system and the reduction of the cost are attained.

key words: *guaranteed cost control, additive gain perturbations, neural networks, LMI*

1. Introduction

In recent years, the problem of robust control for the discrete-time system with parameter uncertainties has been studied (see e.g., [1] and reference therein). In these studies, much effort has been made towards finding a controller that guarantees robust stability. However, it is very important to take into account not only the robust stability but also an adequate cost performance. One approach to this problem is the so-called guaranteed cost control approach [3]. This approach has the advantage of providing an upper bound on a given performance index. The guaranteed cost control for the uncertain discrete-time system by means of the output feedback control has been discussed in [4]. On the other hand, recent advance in theory of Linear Matrix Inequality (LMI) has allowed a revisiting of the guaranteed cost control approach. The guaranteed cost control problem for a class of the uncertain discrete-time system which is based on the LMI design approach was solved by using the state feedback [2]. Very recently, the LMI-based guaranteed cost stabilization for the uncertain discrete-time large-scale systems has been discussed [5], [6]. However, due to the presence of the design parameter of the LMI conditions, it is well-known that the cost becomes quite large.

A neural network (NN) has been actively exploited to construct an intelligent control system because of its nonlinear mapping approximation for the system uncertainties involved. Then some control methodologies utilizing NN have been proposed by combining the modern control theory. For example, the adaptive controller using NN was designed within the framework of the adaptive control theory in the literature [7]. The feedback control systems in which NN were placed instead of a conventional controller [8] or in parallel [9] for identifying and canceling the plant uncertainties have been proposed. As important studies in particular, the Linear Quadratic Regulator (LQR) problem using the multiple NN has been investigated [10]. In these approaches, one neural network is dedicated to the forward model for identifying the uncertainties of the controlled plant, and the other network may compensate for the influence of the uncertainties based on the trained forward model. However, in these researches, there is a possibility that the existing neurocontroller may not stabilize the plant because the stability of the closed-loop system which includes the neurocontroller has not been considered. In fact, it has been shown that the system stability is destroyed when the degree of system nonlinearity is strong [10]. In order to avoid this problem, the stability of the closed-loop system with the neurocontroller has been studied via the LMI-based design approach [14], [15]. However, in these researches, the uncertainty in the input matrix has not been considered.

In this paper, the guaranteed cost control problem of the discrete-time uncertain system that has uncertainty in both state and input matrices is discussed. A class of the fixed state feedback controller of the discrete-time uncertain system with the gain perturbations is newly established by means of the Linear Matrix Inequality (LMI). In order to reduce the large cost caused by the guaranteed cost control, NN is used. A new idea is that the neurocontroller is substituted for the additive gain perturbations. As a result, although the neurocontroller is included in the discrete-time uncertain system, the robust stability of the closed-loop system and the reduction of the cost are attained. It should be noted that there is no result for the stability of the closed-loop system which includes the neurocontroller so far. Finally, in order to demonstrate the efficiency of our design approach, the numerical example is given.

2. Preliminary

Consider the following class of an uncertain discrete-time linear system with additive gain perturbations:

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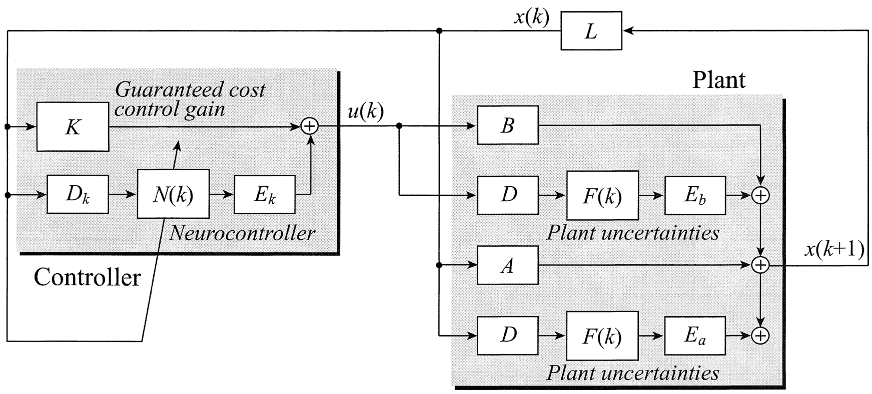

Fig. 1 Block diagram of a new proposed method.

$$x(k+1) = [A + DF(k)E_a]\, x(k) + [B + DF(k)E_b]\, u(k),\ x(0) = x^0, \tag{1a}$$

$$u(k) = [K + D_k N(k) E_k]\, x(k), \tag{1b}$$

where $x(k) \in \Re^n$ is the state, $u(k) \in \Re^m$ is the control input, A, B, D, D_k, E_a, E_b and E_k are known constant matrices, K is the fixed control matrix for the controller (1b), and $F(k) \in \Re^{p_a \times q_a}$ is unknown matrix function and $N(k) \in \Re^{p_n \times q_n}$ is the output of NN. Without loss of generality, it is assumed that $F(k)$ and $N(k)$ satisfy

$$F^T(k)F(k) \le I_{q_a},\ N^T(k)N(k) \le I_{q_n}. \tag{2}$$

These conditions seem to be conservative, while it is possible to establish the LMI condition (8) that will appear later as the sufficient condition.

In this paper, the controlled plant class is an uncertain discrete-time linear system with additive gain perturbations. Namely, the controlled plant class such that the obtained LMI condition holds is considered for the existence of control matrix K. On the other hand, the matching conditions (2) are assumed. It should be noted that $N(k)$ is not systems uncertainty but the output of NN. It should also be noted that these conditions are less conservative under the study of the uncertain discrete-time systems because such conditions are based on the control-oriented assumption that are made in the existing results [1], [2], [4]–[6].

The block diagram of the new proposed method is shown in Fig. 1, where L is a time lag. It should be noted that the controller (1b) has the neurocontroller as the additive gain perturbations $D_k N(k) E_k$ compared to existing method [10].

Associated with the system (1) is the quadratic cost function

$$J = \sum_{k=0}^{\infty} \left[x^T(k)Qx(k) + u^T(k)Ru(k) \right], \tag{3}$$

where Q and R are given as the positive definite symmetric matrices. In this situation, the definition of the guaranteed cost control with the additive gain perturbations is given below.

Definition 1: For the discrete-time uncertain system (1) and the cost function (3), if there exist a control matrix K and a positive scalar J^* such that for the admissible uncertainties and the neurocontroller (2), the closed-loop system is asymptotically stable and the closed-loop value of the cost function (3) satisfies $J < J^*$, then J^* and K are said to be the guaranteed cost and the guaranteed cost control matrix, respectively.

The above definition is very popular for dealing with the time-varying uncertainties and is also used in [3].

It should be noted that if the controller (1b) is the guaranteed cost control, then it is also the quadratically stabilizing controller. Conversely, it can be easily shown that the quadratically stabilizing controller will achieve the guaranteed cost. The following result shows that the guaranteed cost control for the system (1) has the upper bound on the cost function (3).

Lemma 1: Suppose that the following matrix inequality holds for the uncertain discrete-time system (1) with the cost function (3) and for all $x(k) \neq 0$.

$$x^T(k+1)Px(k+1) - x^T(k)Px(k) + x^T(k)[Q + \tilde{K}^T R \tilde{K}]x(k) < 0, \tag{4}$$

where $\tilde{K} := K + D_k N(k) E_k$.

If such condition is met, the matrix K of the controller (1b) is the guaranteed cost control matrix associated with the cost function (3). That is, the closed-loop uncertain system

$$x(k+1) = [(A + DF(k)E_a) + (B + DF(k)E_b) \cdot (K + D_k N(k) E_k)]x(k), \tag{5}$$

is stable and achieves

$$J < J^* = x^T(0)Px(0). \tag{6}$$

Proof: Let us define the following Lyapunov function candidate

$$V(x(k)) = x^T(k)Px(k), \tag{7}$$

where P is the positive definite matrix. Since this proof can

$$\begin{bmatrix} -X & [AX+BY]^T & Y^T & 0 & [E_aX]^T & \left[E_b^TY\right]^T & X \\ AX+BY & -X+(\mu_1+\mu_2)DD^T & BD_kD_k^T & BD_k & 0 & 0 & 0 \\ Y & \left[BD_kD_k^T\right]^T & -R^{-1}+D_kD_k^T & 0 & \left[E_bD_kD_k^T\right]^T & 0 & 0 \\ 0 & [BD_k]^T & 0 & -I_{p_n} & [E_bD_k]^T & 0 & 0 \\ E_aX & 0 & E_bD_kD_k^T & E_bD_k & -\mu_1I_{q_a} & 0 & 0 \\ E_bY & 0 & 0 & 0 & 0 & -\mu_2I_{q_a} & 0 \\ X & 0 & 0 & 0 & 0 & 0 & -(Q+E_k^TE_k)^{-1} \end{bmatrix} < 0. \tag{8}$$

$$\begin{bmatrix} -P & [A+BK]^T & K^T & 0 & E_a^T & \left[E_b^TK\right]^T & I_n \\ A+BK & -P^{-1}+(\mu_1+\mu_2)DD^T & BD_kD_k^T & BD_k & 0 & 0 & 0 \\ K & \left[BD_kD_k^T\right]^T & -R^{-1}+D_kD_k^T & 0 & \left[E_bD_kD_k^T\right]^T & 0 & 0 \\ 0 & [BD_k]^T & 0 & -I_{p_n} & [E_bD_k]^T & 0 & 0 \\ E_a & 0 & E_bD_kD_k^T & E_bD_k & -\mu_1I_{q_a} & 0 & 0 \\ E_bK & 0 & 0 & 0 & 0 & -\mu_2I_{q_a} & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & -(Q+E_k^TE_k)^{-1} \end{bmatrix} < 0,$$

$$\Leftrightarrow L := \begin{bmatrix} -P+Q+E_k^TE_k+\mu_2^{-1}\left[E_b^TK\right]^TE_bK & [A+BK]^T & K^T & 0 & E_a^T \\ A+BK & -P^{-1}+\mu_1DD^T+\mu_2DD^T & BD_kD_k^T & BD_k & 0 \\ K & \left[BD_kD_k^T\right]^T & -R^{-1}+D_kD_k^T & 0 & \left[E_bD_kD_k^T\right]^T \\ 0 & [BD_k]^T & 0 & -I_{p_n} & [E_bD_k]^T \\ E_a & 0 & E_bD_kD_k^T & E_bD_k & -\mu_1I_{q_a} \end{bmatrix} < 0. \tag{9}$$

$$\begin{aligned} M_1 &= \begin{bmatrix} -P+Q+E_k^TE_k & [(A+\Delta A)+(B+\Delta B)K]^T & K^T & 0 \\ (A+\Delta A)+(B+\Delta B)K & -P^{-1} & (B+\Delta B)D_kD_k^T & (B+\Delta B)D_k \\ K & \left[(B+\Delta B)D_kD_k^T\right]^T & -R^{-1}+D_kD_k^T & 0 \\ 0 & [(B+\Delta B)D_k]^T & 0 & -I_{p_n} \end{bmatrix} \\ &= \begin{bmatrix} -P+Q+E_k^TE_k & [A+BK]^T & K^T & 0 \\ A+BK & -P^{-1} & BD_kD_k^T & BD_k \\ K & \left[BD_kD_k^T\right]^T & -R^{-1}+D_kD_k^T & 0 \\ 0 & [BD_k]^T & 0 & -I_{p_n} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ D \\ 0 \\ 0 \end{bmatrix} F(k) \begin{bmatrix} E_a & 0 & E_bD_kD_k^T & E_bD_k \end{bmatrix} + \begin{bmatrix} E_a^T \\ 0 \\ \left[E_bD_kD_k^T\right]^T \\ [E_bD_k]^T \end{bmatrix} F^T(k) \begin{bmatrix} 0 & D^T & 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ D \\ 0 \\ 0 \end{bmatrix} F(k) \begin{bmatrix} E_bK & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} [E_bK]^T \\ 0 \\ 0 \\ 0 \end{bmatrix} F^T(k) \begin{bmatrix} 0 & D^T & 0 & 0 \end{bmatrix} \\ &\le \begin{bmatrix} -P+Q+E_k^TE_k & [A+BK]^T & K^T & 0 \\ A+BK & -P^{-1} & BD_kD_k^T & BD_k \\ K & \left[BD_kD_k^T\right]^T & -R^{-1}+D_kD_k^T & 0 \\ 0 & [BD_k]^T & 0 & -I_{p_n} \end{bmatrix} \\ &\quad +\mu_1 \begin{bmatrix} 0 \\ D \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & D^T & 0 & 0 \end{bmatrix} + \mu_1^{-1} \begin{bmatrix} E_a^T \\ 0 \\ \left[E_bD_kD_k^T\right]^T \\ [E_bD_k]^T \end{bmatrix} \begin{bmatrix} E_a & 0 & E_bD_kD_k^T & E_bD_k \end{bmatrix} \\ &\quad +\mu_2 \begin{bmatrix} 0 \\ D \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & D^T & 0 & 0 \end{bmatrix} + \mu_2^{-1} \begin{bmatrix} [E_bK]^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} E_bK & 0 & 0 & 0 \end{bmatrix} = M_2. \end{aligned} \tag{10}$$

be done by using the similar technique in [11], it is omitted. □

The objective of this section is to design the fixed guaranteed cost control matrix K for the discrete-time system (1).

Theorem 1: Consider the uncertain discrete-time system (1) and cost function (3). For the uncertain matrix $F(k)$ and the output of NN $N(k)$, if the LMI (8) has a feasible solution such as the symmetric positive definite matrix $X \in \Re^{n\times n}$, the matrix $Y \in \Re^{m\times n}$, and the positive scalars $\mu_1 > 0$ and $\mu_2 > 0$, then $K = YX^{-1}$ is the guaranteed cost control matrix.

Furthermore, the corresponding value of the cost func-

$$
\begin{aligned}
N_1 &= \begin{bmatrix} -P+Q & [(A+\Delta A)+(B+\Delta B)(K+\Delta K)]^T & [K+\Delta K]^T \\ (A+\Delta A)+(B+\Delta B)(K+\Delta K) & -P^{-1} & 0 \\ K+\Delta K & 0 & -R^{-1} \end{bmatrix} \\
&= \begin{bmatrix} -P+Q & [(A+\Delta A)+(B+\Delta B)K]^T & K^T \\ (A+\Delta A)+(B+\Delta B)K & -P^{-1} & 0 \\ K & 0 & -R^{-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ (B+\Delta B)D_k \\ D_k \end{bmatrix} N(k) \begin{bmatrix} E_k & 0 & 0 \end{bmatrix} + \begin{bmatrix} E_k^T \\ 0 \\ 0 \end{bmatrix} N^T(k) \begin{bmatrix} 0 & [(B+\Delta B)D_k]^T & D_k^T \end{bmatrix} \\
&\le \begin{bmatrix} -P+Q & [(A+\Delta A)+(B+\Delta B)K]^T & K^T \\ (A+\Delta A)+(B+\Delta B)K & -P^{-1} & 0 \\ K & 0 & -R^{-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} E_k^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} E_k & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ (B+\Delta B)D_k \\ D_k \end{bmatrix} \begin{bmatrix} 0 & [(B+\Delta B)D_k]^T & D_k^T \end{bmatrix} = N_2. \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
N_1 < 0 &\Leftrightarrow \begin{bmatrix} -P+Q+(K+\Delta K)^T R(K+\Delta K) & [(A+\Delta A)+(B+\Delta B)(K+\Delta K)]^T \\ (A+\Delta A)+(B+\Delta B)(K+\Delta K) & -P^{-1} \end{bmatrix} < 0, \\
&\Leftrightarrow [(A+\Delta A)+(B+\Delta B)(K+\Delta K)]^T P [(A+\Delta A)+(B+\Delta B)(K+\Delta K)] \\
&\qquad -P+Q+(K+\Delta K)^T R(K+\Delta K) < 0 \Rightarrow (4). \qquad (12)
\end{aligned}
$$

tion (3) satisfies the following inequality (13) for all admissible uncertainties $F(k)$, and the output of NN $N(k)$

$$J < J^* = x^T(0)X^{-1}x(0). \qquad (13)$$

It should be noted that the matrix Y would be asymmetric.

In order to prove Theorem 1, the following Lemma will be used [12].

Lemma 2: Consider the appropriate matrix $\mathcal{F}$ which is satisfying $\mathcal{F}\mathcal{F}^T \le I_n$ and for any matrices $\mathcal{G}$ and $\mathcal{H}$, there exists the positive parameter $\lambda > 0$ such that the following inequality holds

$$\mathcal{G}\mathcal{F}\mathcal{H} + \mathcal{H}^T\mathcal{F}^T\mathcal{G}^T \le \lambda\mathcal{G}\mathcal{G}^T + \lambda^{-1}\mathcal{H}^T\mathcal{H}. \qquad (14)$$

Let us prove Theorem 1 by using the above Lemma 2.

Proof: Let us introduce the matrices $X = P^{-1}$ and $Y = KP^{-1}$. Pre- and post-multiplying both sides of the inequality (8) by the positive definite matrix

$$\begin{bmatrix} P & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{p_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{q_a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{q_a} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n \end{bmatrix} > 0 \qquad (15)$$

and applying the Schur complement [13] gives (9). It should be noted that the LMI (8) is equivalent to the matrix inequality (9). Using the standard matrix inequality (14) of Lemma 2, for all admissible uncertainty, the matrix inequality (10) holds. Thus, it is easy to verify that $M_1 < 0$ because $L < 0$ is equivalent to $M_2 < 0$ by applying the Schur complement. Moreover, applying the standard matrix inequality (14) of Lemma 2 to N_1 which is defined by the left–hand side of (11) results in the inequality (11). Since $M_1 < 0$ is equivalent to $N_2 < 0$, $N_1 < 0$ holds for all output of NN. Finally, using the matrix inequality (12) which is equivalent to (11), it is concluded that the matrix inequality (4) is satisfied. Thus, K is the guaranteed cost control matrix. On the other hand, since the results of the cost bound (13) can be proved by using the similar argument for the proof of Theorem 1, it is omitted. □

Remark 1: X and Y cannot always be found. Therefore, since Theorem 1 is the sufficient condition, there is the possibility for the existence of other controller that the LMI condition (8) is not satisfied. However, the existence of the robust controller can discriminate by using the LMI condition (8) only. Namely, the existence of the proposed controller is guaranteed as long as the LMI condition (8) holds.

Since the LMI (8) consists of a convex solution set of (μ_1, μ_2, X, Y), various efficient convex optimization algorithm can be applied. Moreover, its solutions represent a set of the guaranteed cost control matrix K. This parameterized representation can be exploited to design the guaranteed cost control gain which minimizes the value of the guaranteed cost for the closed-loop uncertain system. Consequently, solving the following optimization problem allows us to determine the optimal bound.

$$J < J^* < \min_{(\mu_1,\ \mu_2,\ X,\ Y)} \alpha, \qquad (16)$$

such that (8) and

$$\begin{bmatrix} -\alpha & x^T(0) \\ x(0) & -X \end{bmatrix} < 0. \qquad (17)$$

The problem addressed in this section is defined as follows:

Problem 1: Find the guaranteed cost control matrix $K = YX^{-1}$ such that the LMIs (8) and (17) are satisfied and the cost α become as small as possible.

The bound in *Problem 1* depends on the initial condition $x(0)$. To remove this dependence on $x(0)$, it is assumed that $x(0)$ is a zero mean random variable satisfying $E[x(0)x^T(0)] = I_n$. Then, the LMI (17) implies

$$\begin{bmatrix} -M & I_n \\ I_n & -X \end{bmatrix} < 0, \tag{18}$$

where $E[\cdot]$ denotes the expectation, M is the expectation of α.

The crucial difference between the uncertain discrete-time system in [1], [2] and the considered system of this paper is that the controller gain perturbations as the neurocontroller are newly added. Furthermore, the LMI approach has been newly applied to the guaranteed cost control problem for the discrete-time system that includes the uncertainty in both state and input matrices compared to the existing results [14], [15]. Therefore, the obtaining results of this section are original.

3. Neural Networks for Additive Gain Perturbations

The LMI design approach usually yields the conservative controller due to the presence of the uncertainty $F(k)$ and the additive gain perturbations $N(k)$ as the output of NN. The main purpose of this paper is to introduce NN as additive gain perturbations into the discrete-time uncertain system to improve the cost performance. It should be noted that the proposed neurocontroller regulates its outputs in real-time under the robust stability guaranteed by the LMI approach.

3.1 On-Line Learning Algorithm of Neurocontroller

It is expected that the reduction of the cost will be attained when the neurocontroller can manage the uncertain system as the nominal linear system while compensating for control errors by the conservative controller. That is, the neurocontroller is required to compensate the conservative controller to work as the LQR controller in the uncertain system.

Let us consider the following nominal system without the uncertainty and the gain perturbation.

$$\hat{x}(k+1) = A\hat{x}(k) + B\hat{u}(k), \tag{19}$$

where $\hat{x}(k) \in \Re^n$ is the state and $\hat{u}(k) \in \Re^m$ is the control input. For such linear system, it is well-known that the LQR control is an effective method to design the controller which can minimize the cost function (3). Based on the LQR, the optimal control $\hat{u}^*(k)$ can be designed as

$$\hat{u}^*(k) = \hat{K}\hat{x}(k), \tag{20a}$$

$$\hat{K} = -(R + B^T\hat{P}B)^{-1}B^T\hat{P}A, \tag{20b}$$

where $\hat{K}$ is the optimal feedback gain matrix and the matrix $\hat{P}$ is the positive semidefinite symmetric solution of the following algebraic Riccati equation.

$$\hat{P} = A^T\hat{P}A - A^T\hat{P}B(R + B^T\hat{P}B)^{-1}B^T\hat{P}A + Q. \tag{21}$$

It is known that the guaranteed cost of the nominal system is smaller than that of the uncertain system. It can be much expected that the reduction of the cost will be attained when the response of the uncertain system behaves like that of the nominal system. Therefore, the NN of the proposed system is trained at the real-time so that the state discrepancy $\|\hat{x}(k+1)-x(k+1)\|$ becomes as small as possible at each step k. An energy function $E(k)$ is defined as the discrepancy between the behavior of the nominal system according to the LQR method and the one of the uncertain discrete-time system of step k. At each step, the weight coefficients are modified so as to minimize $E(k)$ which is given by

$$E(k) := \frac{1}{2}\left(\hat{x}(k+1) - x(k+1)\right)^T\left(\hat{x}(k+1) - x(k+1)\right). \tag{22}$$

If $E(k)$ can be minimized as small as possible, the discrepancy $\|\hat{x}(k+1) - x(k+1)\|^2$ would be also minimized so that the cost of the uncertain discrete-time system is close to the cost of the nominal system based on the LQR control.

In the learning phase of NN, the weight updating rules can be described as

$$w_g^{ij}(k+1) = w_g^{ij}(k) + \Delta w_g^{ij}(k). \tag{23}$$

On the other hand, the modification of the weight coefficient $w_g^{ij}(k)$ is given by

$$\Delta w_g^{ij}(k) = -\varepsilon\frac{\partial E(k)}{\partial w_g^{ij}(k)}, \tag{24a}$$

$$\frac{\partial E(k)}{\partial w_g^{ij}(k)} = \frac{\partial E(k)}{\partial N(k)} \cdot \frac{\partial N(k)}{\partial w_g^{ij}(k)}, \tag{24b}$$

where ε is the learning ratio.

The term $\frac{\partial E(t)}{\partial N(k)}$ of the equation (24b) can be calculated from the energy function (22) as follows:

$$\frac{\partial E(k)}{\partial N(k)} = -\left(\hat{x}(k+1) - x(k+1)\right)(B + DF(k)E_b)D_kE_k. \tag{25}$$

Since the variable $F(k)$ is unknown matrix function, the equation (25) that is used to learn for NN would not be calculated. In order to remove this problem, suppose there exists a parameter $\Gamma(k)$ such that $B + DF(k)E_b \approx \Gamma(k)B$. It should be noted that $\Gamma(k)$ is the matrix value function and its elements are the positive scalar. It is worth pointing out that even though the above assumption is conservative condition, it is compensated by making use of learning ratio.

Then, the above equation (25) can be rewritten as follows:

$$\frac{\partial E(k)}{\partial N(k)} \approx -\left(\hat{x}(k+1) - x(k+1)\right)\Gamma(k)BD_kE_k. \tag{26}$$

However, because of $\Gamma(k)$, it is difficult to decide the learning rule of NN. Hence, it is necessary to set ε according to $\Gamma(k)$. In this paper, the modification of the weight coefficient of the equation (24a) is defined as follows:

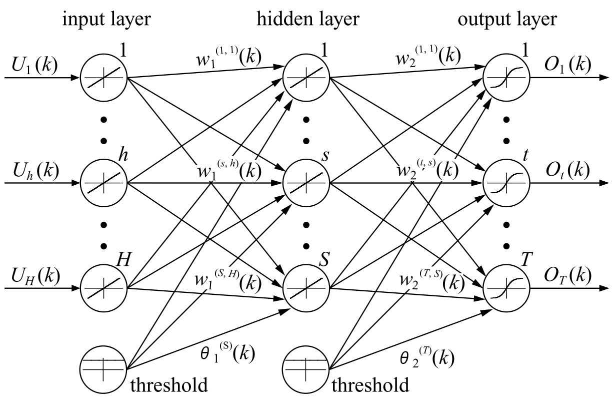

Fig. 2 Structure of the multilayered neural networks.

$$\Delta w_g^{ij}(k) \approx \eta\,(\hat{x}(k+1) - x(k+1))BD_kE_k\frac{\partial N(k)}{\partial w_g^{ij}(k)}, \tag{27}$$

where $\eta := \varepsilon|\Gamma(k)|$ is defined as a new learning ratio. η is used instead of deciding ε according to $\Gamma(k)$. $\dfrac{\partial N(k)}{\partial w_g^{ij}(k)}$ can be calculated using the chain rule on the NN. From (23) and (27), NN can be trained so as to decrease the cost J on-line.

3.2 Multilayered Neural Networks

The utilized NN are of a three-layer feed-forward network as shown in Fig. 2. The linear function is utilized in the neurons of the input and the hidden layers, and a sigmoid function in the output layer. The inputs and outputs of each layer can be described as follows.

$$s_g^i(k) := \begin{cases} U_i(k) & \{g = 1(\text{input layer})\} \\ \sum w_1^{(i,j)}(k)o_1^j(k) & \{g = 2(\text{hidden layer})\} \\ \sum w_2^{(i,j)}(k)o_2^j(k) & \{g = 3(\text{output layer})\}, \end{cases}$$

$$o_g^i(k) := \begin{cases} s_1^i(k) & \{g = 1(\text{input layer})\} \\ s_2^i(k) + \theta_1^{(i)}(k) & \{g = 2(\text{hidden layer})\} \\ \dfrac{1 - e^{(-s_3^i(k)+\theta_2^{(i)}(k))}}{1 + e^{(-s_3^i(k)+\theta_2^{(i)}(k))}} & \{g = 3(\text{output layer})\}, \end{cases}$$

where $s_g^i(k)$ and $o_g^i(k)$ are the input and the output of the neuron i in the gth layer at the step k. $w_g^{i,j}(k)$ indicates the weight coefficient from the neuron j in the gth layer to the neuron i in the $(g+1)$th layer. $U_i(k)$ is the input of NN. $\theta_g^{(i)}(k)$ is a positive constant for the threshold of the neuron i in the $(g + 1)$th layer. As the additive gain perturbations defined in the formula (2), the outputs of NN are set in the range of $[-1.0,\ 1.0]$.

4. Numerical Example

In order to demonstrate the effectiveness of our proposed method, a numerical example is given. The system matrices are as follows.

$$A = \begin{bmatrix} -0.2 & 1.0 \\ 1.0 & -0.1 \end{bmatrix},\ B = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix},\ D = \begin{bmatrix} 0.2 \\ 0.0 \end{bmatrix},$$

$$E_a = \begin{bmatrix} 2.0 & 2.2 \end{bmatrix},\ E_b = 0.1,\ D_k = \begin{bmatrix} 0.1 & 0.13 \end{bmatrix},$$

$$E_k = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},\ F(k) = f(k),$$

$$N(k) = \begin{bmatrix} N_1(k) & 0 \\ 0 & N_2(k) \end{bmatrix},$$

where $N_1(k)$ and $N_2(k)$ are the outputs of NN. The initial system condition is $x(0) = [4\ \ 4]^T$, and the weighting matrices are chosen as $Q = \text{diag}(1,\ 2)$ and $R = 1$, respectively. It should be noted that for LQR theory, the parameters Q and R are chosen freely by the control system designer without any constraint.

Solving the LMI (8), the solutions are given by

$$X = \begin{bmatrix} 8.4462e-02 & -3.0146e-02 \\ -3.0146e-02 & 1.3228e-01 \end{bmatrix},$$

$$Y = \begin{bmatrix} 2.6144e-02 & -1.2549e-01 \end{bmatrix},$$

$$\mu_1 = 1.2025, \quad \mu_2 = 4.6081e-02.$$

Since $K := YX^{-1}$, the state feedback control gain K which is based on the proposed LMI design method with the neurocontroller is given by

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} -3.1618e-02 & -9.5584e-01 \end{bmatrix}. \tag{28}$$

Consequently, the optimal guaranteed cost of the uncertain discrete-time closed-loop system which is given by (6) is $J^* = 4.3167e+02$.

Based on the LQR control which is constructed from (20) and (21), the state feedback control gain $\hat{K}$ is calculated as follows:

$$\hat{K} = \begin{bmatrix} \hat{K}_1 & \hat{K}_2 \end{bmatrix} = \begin{bmatrix} 2.7267e-01 & -8.0558e-01 \end{bmatrix}. \tag{29}$$

For the system without the proposed neurocontroller, that is, $N(k) \equiv 0$, the control input of the uncertain system is described by

$$u(k) = \bar{K}x(k), \tag{30}$$

where the state feedback control gain $\bar{K}$ is designed by using the LMI approach which is proposed in [2].

$$\bar{K} = \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix} = \begin{bmatrix} 1.1722e-01 & -9.9172e-01 \end{bmatrix}. \tag{31}$$

The neurocontroller is composed of 30 neurons in the hidden layer and two neurons in the input and the output layers, respectively. The initial weights are set randomly in the range of $[-0.05,\ 0.05]$. Various uncertain systems were examined for $f(k) = 1$, $f(k) = \exp(-0.5k)$ and $f(k) = \cos(1/18\pi k)$. Table 1 shows that the cost of the proposed system is smaller than that of the system without the

Table 1 A comparison of the cost in each condition.

The cost of the nominal system $\hat{J}$ = 8.9562e+01

$f(k)$	η	With NN	Without NN
1	0.6	1.4085e+02	1.7810e+02
$\exp(-0.5k)$	0.1	1.2625e+02	1.3203e+02
$\cos(1/18\pi k)$	0.6	1.3877e+02	1.6243e+02

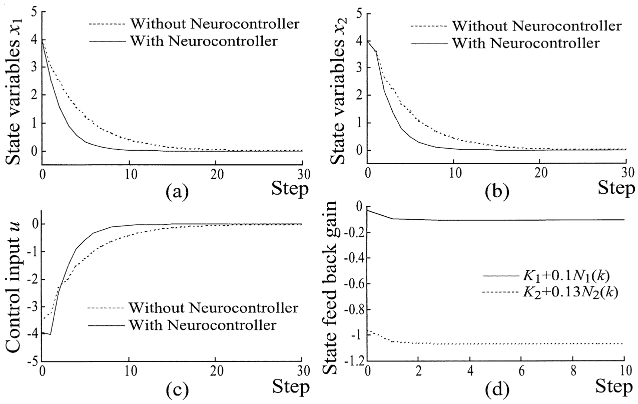

Fig. 3 Response of the closed-loop system with the uncertainty $f(k) = 1$ and the neurocontroller. (a), (b) State variables. (c) Control input. (d) State feedback gain with additive gain.

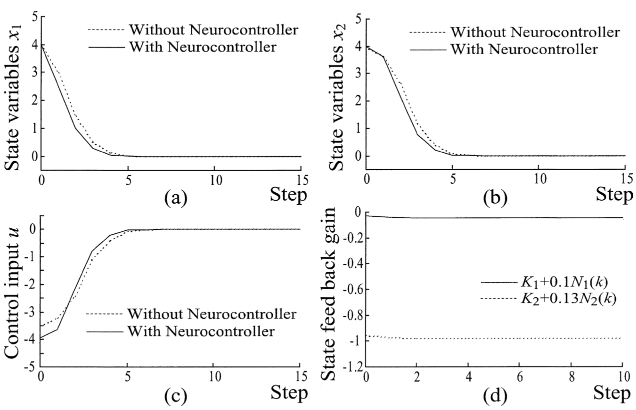

Fig. 4 Response of the closed-loop system with the uncertainty $f(k) = \exp(-0.5k)$ and the neurocontroller. (a), (b) State variables. (c) Control input. (d) State feedback gain with additive gain.

neurocontroller in all cases. Therefore, it is shown from Table 1 that the new proposed neurocontroller succeeds in improving the cost.

The simulation results ($f(k) = 1$) are shown in Fig. 3. The response of the proposed neurocontroller is stabilized faster than the controller without the neurocontroller (Fig. 3 (a), (b) and (c)) is used. Figure 3 (d) shows the result for the feedback gain with the additive gain $\tilde{K}$, i.e., $K + D_k N(k) E_k$. As another example, the simulation results are shown in Figs. 4, 5. The response of the proposed method is also stabilized faster than the controller without NN is used. The proposed neurocontroller could reduce the cost and compensate for the uncertainties of each system.

Figure 6 shows the response with the proposed neurocontroller and the LQR control ($f(k) = 1$). The state vari-

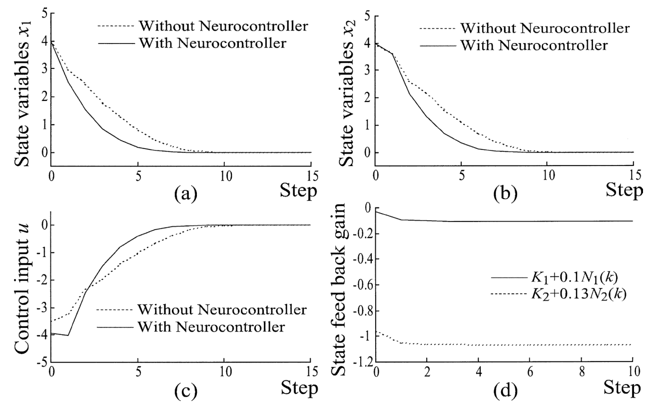

Fig. 5 Response of the closed-loop system with the uncertainty $f(k) = \cos(1/18\pi k)$ and the neurocontroller. (a), (b) State variables. (c) Control input. (d) State feedback gain with additive gain.

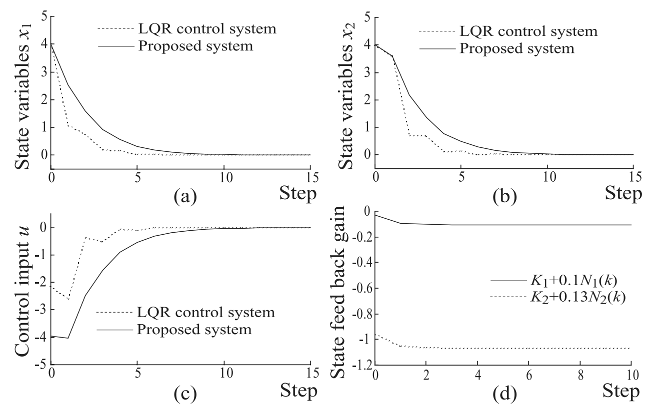

Fig. 6 Response of the closed-loop system for the neurocontroller versus LQR control under the uncertainty $f(k) = 1$. (a), (b) State variables. (c) Control input. (d) State feedback gain with additive gain.

ables x_1 and x_2 can trace the state variables $\hat{x}_1$ and $\hat{x}_2$ well as shown in Fig. 6 (a), (b). Since it is shown from Fig. 6 (d) that $\tilde{K}$ changes for compensating for the system uncertainties and its response can be close to the nominal response via the LQR design method, the proposed neurocontroller succeeds in reducing the cost. Therefore, the energy function $E(k)$ is adequate for the learning algorithm.

5. Conclusions

The application of neural networks to the guaranteed cost control problem of the discrete-time system that has uncertainties in both state and input matrices has been investigated. Although the results presented seem to be slight modification of the existing results [14] in the sense that the systems uncertainties are included in both state and input matrices, the new LMI condition and the learning algorithm of the NN have been derived. Particularly, it has succeeded in avoiding the Bilinear Matrix Inequality (BMI) condition that has been established in [16]. Substituting the neurocontroller into the gain perturbations, the robust stability of the closed-loop system is guaranteed even if the systems include

NN. Moreover, the reduction of the cost is attained by using the neurocontroller. The numerical example have shown the excellent result that the NN has succeeded in reducing the large cost caused by the LMI.

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