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INTERNAL FORCES AND STABILITY OF MULTI-FINGER GRASPS

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Abstract. This paper deals with the rotational stability of a rigid body under the constant contact forces. For this problem, first, the stiffness tensor is constructed and its basic properties are analyzed. Stability due to the gravitational and the internal forces is considered separately. For the gravity-induced stiffness only one necessary condition of stability, formulated in terms of geometric and gravity centers, is obtained. The internal force parameterization is done with the use of a virtual linkage/spring model. Within this parameterization, necessary and sufficient conditions of stability under internal force loading are obtained in the analytical form. In the space of the internal forces they form a region given by intersection of a plane and a singular quadric. These conditions can be incorporated into the grasping force planner.

Key Words. Multi-finger grasp; rotational stability; stiffness tensor; internal forces

1. INTRODUCTION

One of the fundamental problems in controlling multi-fingered hands is stability of the resulting grasp. In recent years, the problem has been addressed from different points of view and a number of approaches to defining the grasping stability, its robustness, and relation to such concepts as grasping form and force closure, has been proposed in literature. A very good survey on this topic can be found in (Shimoga, 1996). Here in this paper we address the problem in a somewhat simplified way, dealing only with the rotational stability of the object.

Basically, the total compliance of the object, grasped by multiple arms or fingers, C_{object},has two sources:

$$C_{object} = C_{fingers} + C_{loading}. \tag{1}$$

The first one is due to compliance of the arm/finger itself, and the second one is due to the contact force interaction between the arm and the robot. Roughly, the first term is defined by the transformation of the joint compliance C_{joint} to the Cartesina level through the arm jacobian J. The Cartesian compliance of the arm, $C_{fingers} = JC_{joints}J^{T}$, is symmetric and positive definite (and therefore stable) as long as the joint compliance matrix is stable. The compliance due to the finger interaction, $C_{loading}$, is not necessarily positive definite. It depends on the contact force distribution, and is often the source of grasping instability. This phenomena has been discovered and studied by (Nguen, 1989; Cutkosky and I.Kao, 1989; Kaneko *et al.*, 1990). It should be noted that very similar subject—stability due

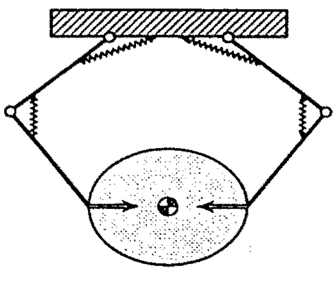

Fig. 1. Sources of the object compliance.

to internal forces in mechanisms with closed kinematic chains—was analyzed by (Hanafusa and Adli, 1991; Yi *et al.*, 1991).

One possible approach to provide stable grasping can be formulated as follows: for a given matrix $C_{loading}$ find out the the total finger compliance $C_{fingers}$ so that the object compliance matrix C_{object} be positive definite. Theoretically this approach can work nicely. However, it is case-dependent approach and there is no systematic procedure for adjusting the compliance of the fingers to that of the object. Another possible solution of the stability problem is based on decomposition of the total compliance and designing the corresponding matrices, $C_{fingers}$ and $C_{loading}$ separately. Indeed, if they are both stable than the resulting compliance will also be stable. Conceptually, this approach is taken in this paper, and in our work we do not consider the compliance of the fingers at all. In so doing, we consider the grasp to be stable if its stiffness matrix is not negative definite. The reason for taking this view is simple—even if the contact force induced compliance is positive semidefinite, the resulting compliance of the system can be easily made stable with simple proportional control of the fingers' joints. In a sense, we shift the problem of stable grasping from control level to the planning level.

Our work was motivated mainly by papers of (Li and Kao, 1995), where fundamental properties of the grasp stiffness matrix were under investigation, and of (Jen *et al.*, 1996), where the rotational stability of the grasped object was analyzed from the classical standpoints of the Lyapunov's theory. However, the relation of the stability to the internal force distribution has not been studied in detail, and necessary and sufficient conditions of stability have not been obtained in the analytical form. In our work we want to fill the gap between the two cited papers. Another papers that are worth noting in this introduction are (Howard and Kumar, 1994), where the geometric properties of the object has been brought into the stability analysis, and (Mason *et al.*, 1995), where gravitational effects in the stability problem has been investigated.

This paper is organized as follows. Firstly, in Section 2 the analytical expression for the stiffness tensor is derived and its basic properties are discussed. Force parameterization for the so-called virtual linkage/spring model is given in Section 3. Stability of the gravitation force loading is discussed in Section 4.1. Necessary and sufficient conditions of positive definiteness of the stiffness tensor under internal force loading are derived in Section 4.2. Finally, conclusions are presented in Section 5.

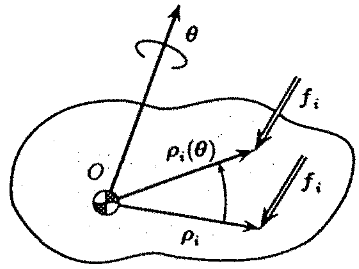

Fig. 2. System notations.

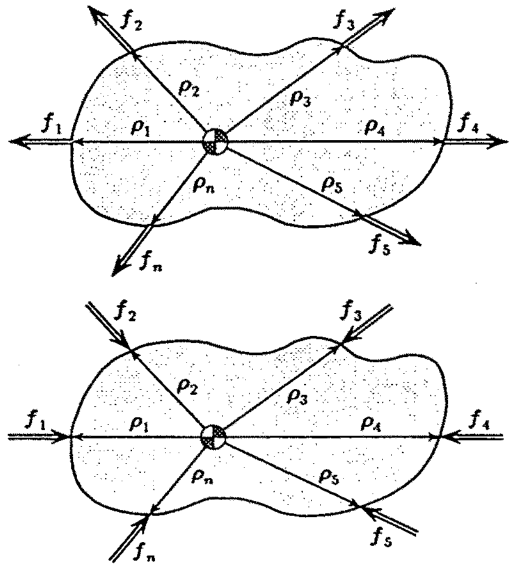

Fig. 3. Stretching and compressive forces.

2. STIFFNESS TENSOR

Let us consider a rigid body subjected to multiple frictional contacts. Assume that the constant forces $f_1, f_2, \ldots, f_n$ are applied at the points defined by the radius-vectors $\rho_1, \rho_2, \ldots, \rho_n$ drawn from the center of mass O. The body is at the equilibrium so the static equations read

$$\sum_{i=1}^{n} f_i = -mg, \quad \sum_{i=1}^{n} \rho_i \times f_i = 0, \tag{2}$$

where m is the mass of the body and g is the gravity vector. Let θ (Fig. 2) be the vector of the finite rotation of the body. The coordinates of the contact points after rotation of the body, $\rho_i(\theta)$, are defined by the Rodrigues' equation for the finite rotation:

$$\rho_i(\theta) = \rho_i + \frac{1}{1+\frac{1}{4}\theta^{\mathrm{T}}\theta}\,\theta \times (\rho_i + \frac{1}{2}\theta \times \rho_i). \tag{3}$$

Since the contact forces are assumed to be constant, calculating the potential energy leads to the following expression:

$$\Pi = -\sum_{i=1}^{n} f_i^{\mathrm{T}} \rho_i(\theta) = \frac{1}{2}\,\frac{1}{1+\frac{1}{4}\theta^{\mathrm{T}}\theta}\,\theta^{\mathrm{T}} K \theta, \tag{4}$$

where

$$K = \sum_{i=1}^{n} (\rho_i^{\mathrm{T}} f_i) I - \rho_i f_i^{\mathrm{T}} \tag{5}$$

is the stiffness tensor. The potential energy is always positive (and the equilibrium is stable) as long as K is positive definite. Note that for the small rotation θ the potential energy is transformed to the familiar quadratic form. Also note that in the planar case the rotational stiffness is a scalar given by

$$K = \sum_{i=1}^{n} \rho_i^{\mathrm{T}} f_i. \tag{6}$$

The following properties of the stiffness tensor can be formulated straight away. First, K is symmetric as long as the object is at equilibrium. Indeed, one can prove that $K - K^{\mathrm{T}} = \Omega(\sum_{i=1}^{n} \rho_i \times f_i)$, and, therefore, the skew-symmetric part of K is always zero by the static equations (2). Here, Ω is the skew-symmetric operator such that $\Omega(a) \cdot b \equiv a \times b$. If, however, the object is not at equilibrium, K is always asymmetric.

Next, even though K is symmetric at the equilibrium, it is not always and not necessarily positive definite. It is only in some simple cases when the

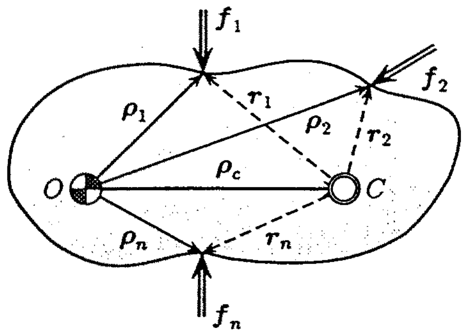

Fig. 4. Shift to the geometric center.

judgment on the positive definiteness of K can be done easily. Consider, for example, the case when all the applied forces are coplanar to the correspondent vectors ρ_i, as shown in Fig. 3. In this case $f_i = k_i\rho_i$, and formula (5) gives

$$K = \sum_{i=1}^{n} k_i\{(\rho_i^{\mathrm{T}}\rho_i)I - \rho_i\rho_i^{\mathrm{T}}\}. \tag{7}$$

As can be easily seen, K has the structure of the inertia tensor of a system of points built on the vectors ρ_i, with k_i playing the role of masses. Therefore, if all $k_i \geq 0$, i.e., all the forces are stretching, K is positive definite and the equilibrium is stable. In the opposite case when all $k_i \leq 0$, i.e., all the forces are compressive, K is negative definite and the equilibrium is unstable.

However, in the general case when k_i have different signs or if the applied forces f_i are not coplanar to ρ_i, it is not that easy to to make a judgment on the properties of K without its direct computing, and additional study of the force structure is required. Finally, please note that the forces we deal with in this paper are assumed to be constant in the inertial frame. If they are constant in the body frame, we can show that $K = \Omega(\sum_{i=1}^{n} \rho_i \times f_i)$, and it is zero by the static equations. In such a case the applied forces f_i do not contribute to the rotational stiffness tensor as long as the body is in equilibrium.

3. FORCE DECOMPOSITION

To relate the stability properties to the structure of the applied forces one should make the force decomposition and obtain analytical solution of the static equations. One possible decomposition can be based on the pseudo-inversion of the grasp matrix. Such decomposition, interpreted in terms of the screw theory, has been given in (Kumar and Waldron, 1989). Here, we present slightly different computational procedure, based on the classical vectorial notation and changing the reference point.

First, in order to make all the calculations easier, we shift the reference point to the geometric center C (Fig. 4) defined by

$$\rho_c = \frac{1}{n}\sum_{i=1}^{n} \rho_i. \tag{8}$$

Introducing the block vectors $\Phi_o = \{-mg, 0\}^{\mathrm{T}}$ and $f = \{f_1^{\mathrm{T}}, \ldots, f_n^{\mathrm{T}}\}^{\mathrm{T}}$, we can rewrite the static equations in the following form:

$$\Phi_o = B_o f = B_{oc}B_c f, \tag{9}$$

where

$$B_{oc} = \left\| \begin{matrix} I & O \\ \Omega(\rho_c) & I \end{matrix} \right\|, \quad B_c = \left\| \begin{matrix} I & \ldots & I \\ \Omega(r_1) & \ldots & \Omega(r_n) \end{matrix} \right\|,$$

and $r_i = \rho_i - \rho_c$. If $n \geq 3$ ($n \geq 2$ in the planar case) and the contact points are not coplanar, $B_{oc}B_c$ is a full-rank decomposition of the grasp matrix B_o and, therefore, $B_o^+ = B_c^+ B_{oc}^{-1}$.

Note that the symbolic computation of the pseudoinverse $B_c^+ = B_c^{\mathrm{T}}(B_c B_c^{\mathrm{T}})^{-1}$ is much easier than that for the original, non-decomposed matrix B_o. It is due to the fact that $\sum_{i=1}^{n} r_i = 0$, and therefore, $B_c B_c^{\mathrm{T}} = \mathrm{diag}\{nI, J_c\}$ has the block-diagonal structure. Here,

$$J_c = \sum_{i=1}^{n} \Omega^{\mathrm{T}}(r_i)\Omega(r_i) \tag{10}$$

is the inertia tensor of the system formed by the points r_i with unit masses. In the general spatial case $J_c \in \Re^{3\times3}$ is symmetric and positive definite if $n \geq 3$ and all points do not lie on a common line.

The general solution of (9) has the following form:

$$f = B_o^+\Phi_o + P_I\varphi, \tag{11}$$

where $P_I = I - B_c^+ B_c$ is the orthogonal projector onto the null space of the grasp matrix B_o, i.e., onto the space of the internal forces, and it does not depend on the reference point ρ_c. Here, in (11), $\varphi = \{\varphi_1^{\mathrm{T}}, \ldots, \varphi_n^{\mathrm{T}}\}^{\mathrm{T}}$ is composed of the arbitrary specified vectors φ_i. Computing (11) in the vectorial form and skipping all the intermediate calculations, we finally arrive at the analytical solution of the static equations. It is given by the orthogonal decomposition of the applied forces, $f = f_G + f_I$, where the gravity-inducing, f_{Gi}, and the internal force, f_{Ii}, components are defined as follows:

$$f_{Gi} = (\Omega^{\mathrm{T}}(r_i)J_c^{-1}\Omega(\rho_c) - \frac{1}{n}I)mg, \tag{12}$$

$$f_{Ii} = \varphi_i - \sum_{k=1}^{n}\{\Omega^{\mathrm{T}}(r_i)J_c^{-1}\Omega(r_k) + \frac{1}{n}I\}\varphi_k. \tag{13}$$

Note that φ defines redundant representation of the internal forces and does not have clear physical meaning. To introduce physically meaningful parameterization of the internal forces, let us, following to (Kumar and Waldron, 1989), characterize the interaction between any two fingers by

$$\alpha_{ij} = (r_i - r_j)^{\mathrm{T}}(f_i - f_j), \tag{14}$$

i.e., by the difference of the contact forces projected along the line joining the two contact points. The interaction force is of compression type if $\alpha_{ij} < 0$, and of tension type if $\alpha_{ij} > 0$. The physical meaning of α_{ij} is the work produced by $f_{ij} = f_i - f_j$ on the displacement $r_{ij} = r_i - r_j$.

Note that the dimension of α_{ij}, [N · m], can also be interpreted as that of the rotational stiffness. Continuing this thought, we could have introduced another possible parameterization of the internal forces by $\tilde{\alpha}_{ij} = \alpha_{ij}/(r_{ij}^{\mathrm{T}} r_{ij})$, where $\tilde{\alpha}_{ij}$ are the stiffness of the linear virtual spring connecting the two contact points. It is remarkable that in this interpretation the grasp of the rigid body can be represented by the virtual springs which can have as positive as well as negative shiftiness. It should also be noted that this interpretation is closed conceptually to the virtual linkage model considered in (Williams and Khatib, 1992). In the forthcoming analysis, for the sake of simplicity of the resulting mathematical expressions, we, however, will deal with the parameterization given by (14).

In the non-redundant, minimal representation of the internal forces, for which there exists one-to-one mapping between the applied forces f and the vector combined of Φ_o and α, the solution of the static equations is specified as

$$f = B_o^{+}\Phi_o + P_{I\alpha}\alpha, \tag{15}$$

where $P_{I\alpha} \in \Re^{2n\times(2n-3)}$ in the planar case, and $P_{I\alpha} \in \Re^{3n\times 3(n-2)}$ in the spatial case. Note that $\alpha = \{\alpha_{ij}\} \in \Re^{\frac{n(n-1)}{2}}$. Equating the dimension of α to the column-dimension of $P_{I\alpha}$, one obtains the numbers of the contact points admissible for the minimal representation. They are $n = 2$ or $n = 3$ in the planar case, and $n = 3$ or $n = 4$ in the spatial case.

To obtain an analytical expression for the matrix $P_{I\alpha}$, one must represent (14) in the matrix form so that

$$\alpha = A_o(r_{ij})f. \tag{16}$$

This representation depends on how the elements of α are ordered. For example, for the three-fingered grasp shown in Fig. 5 with $\alpha = \{\alpha_{23}, \alpha_{31}, \alpha_{12}\}^{\mathrm{T}}$, the matrix A_o can be constructed as follows:

$$A_o = \left\| \begin{array}{ccc} 0 & r_{23}^{\mathrm{T}} & -r_{23}^{\mathrm{T}} \\ -r_{31}^{\mathrm{T}} & 0 & r_{31}^{\mathrm{T}} \\ r_{12}^{\mathrm{T}} & -r_{12}^{\mathrm{T}} & 0 \end{array} \right\|. \tag{17}$$

Now, having constructed A_o, we can prove that the matrix $P_{I\alpha}$ can be expressed in the following form:

$$P_{I\alpha} = A_o^{+} = A_o^{\mathrm{T}}(A_o A_o^{\mathrm{T}})^{-1}. \tag{18}$$

This formula and the representation (15) also remain true for $n \leq 7$, i.e., even though parameterization of the internal forces in terms of α becomes redundant. It, however, still keeps the advantage of having clear physical meaning, and is by no means worse than the parameterization by φ. Note that for $n > 7$ the matrix $A_o A_o^{\mathrm{T}}$ becomes singular and, therefore, representation (18) does not hold.

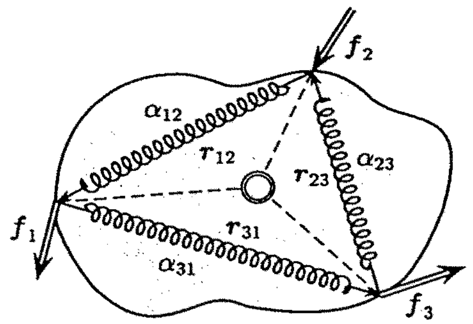

Fig. 5. Virtual springs in three-fingered grasp.

4. STABILITY ANALYSIS

Similarly to the applied forces f, the stiffness tensor K can be decomposed into its gravity-inducing, K_G, and internal force inducing, K_I, components with

$$K = K_G + K_I. \tag{19}$$

In this section, we try to find out condition under which the matrices K_G and K_I are positive semidefinite.

4.1. *Gravity-Induced Stiffness*

We start our analysis with obtaining the analytical expression for K_G. Substituting (12) into (5), after some simplifications we get

$$K_G = \Omega^{\mathrm{T}}(f_c)\Omega(\rho_c) + \sum_{i=1}^{n} r_i \mu_c^{\mathrm{T}} \Omega^{\mathrm{T}}(r_i), \tag{20}$$

where $\mu_c = J_c^{-1} m_c$, $m_c = -\rho_c \times f_c$, and $f_c = -mg$. Next, making use of the Jabobian identity, we can represent K_G through the geometric invariants of the body. This is given by

$$K_G = \Omega^{\mathrm{T}}(\rho_c)\Omega(f_c) + \Omega(\mu_c)\{J_c - \sigma I\}, \tag{21}$$

where $\sigma = \frac{1}{2}\mathrm{tr}J_c$.

If the geometric center of the object coincides with its center of mass ($\rho_c = 0$), K_G does not contribute to the total stiffness. Another particular case of all-zero eigenvalues of the matrix K_G is the one where the object is planar and its plane is orthogonal to the gravity force.

In the planar case $K_G = \rho_c^{\mathrm{T}} f_c$, and the stability condition is

$$K_G = \rho_c^{\mathrm{T}} f_c \geq 0, \tag{22}$$

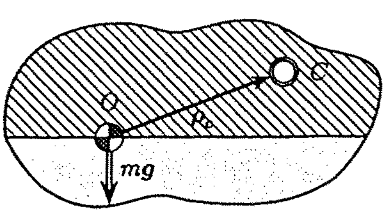

Fig. 6. Gravitational stability.

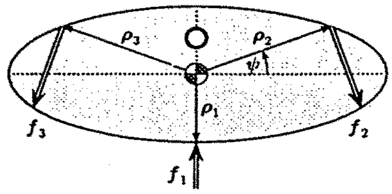

Fig. 7. Symmetric three-fingered grasp.

It has the following geometric interpretation: for the no-internal-force grasp to be stable the center of mass of the object must lie below its geometric center as shown in Fig. 6. For example, in the three-fingered grasp shown in Fig. 7 (2nd and 3rd grasping points are symmetric), the grasping angle ψ should be more than $\pi/6$.

Unfortunately, in the spatial case the judgment on $\boldsymbol{K}_G$ is not that simple. What is necessary and sufficient for the planar case is only necessary for the spatial one. Indeed, inspecting the structure of $\boldsymbol{K}_G$, we can show that $\mathrm{tr}\boldsymbol{K}_G = 2\boldsymbol{\rho}_c^{\mathrm{T}}\boldsymbol{f}_c$ and hence, by the Routh-Hurwicz criterion, the condition (22) is also necessary for the stability in the spatial case. However, taken alone it is not sufficient since the other two conditions of positive definiteness of $\boldsymbol{K}_G$ must be established and tested.

It remains the topic for our future study to find the geometric interpretation—the one given in terms of the geometric invariants of the body—of the full set of stability conditions for the matrix $\boldsymbol{K}_G$. Here, we want to consider two special case when the structure of the eigenvalues of $\boldsymbol{K}_G$ can be identified easily.

In the first particular case the vectors $\boldsymbol{\rho}_c$ and $\boldsymbol{f}_c$ are coplanar, i.e., $\boldsymbol{\rho}_c = k\boldsymbol{f}_c$. Under this assumption the second term in (20) vanishes and it can be shown that the eigenvalues of $\boldsymbol{K}_G$ are as follows: $\lambda_1 = 0, \lambda_{2,3} = \boldsymbol{\rho}_c^{\mathrm{T}}\boldsymbol{f}_c$. Hence, (22) is the only condition for the stability judgment. As such, it is necessary and sufficient condition.

In the second particular case, the matrix product $\boldsymbol{\Omega}(\boldsymbol{\mu}_c)\boldsymbol{J}_c$ is commutative. It holds when, for example, the grasping points form a regular polyhedron (Fig. 8). To make the stability judgment, we represent $\boldsymbol{K}_G$ by the sum of two symmetric matrices $\boldsymbol{K}'_G$ and $\boldsymbol{K}''_G$, which can be constructed in the following form:

$$\boldsymbol{K}'_G = \frac{\boldsymbol{\Omega}^{\mathrm{T}}(\boldsymbol{\rho}_c)\boldsymbol{\Omega}(\boldsymbol{f}_c) + \boldsymbol{\Omega}^{\mathrm{T}}(\boldsymbol{f}_c)\boldsymbol{\Omega}(\boldsymbol{\rho}_c)}{2}, \tag{23}$$

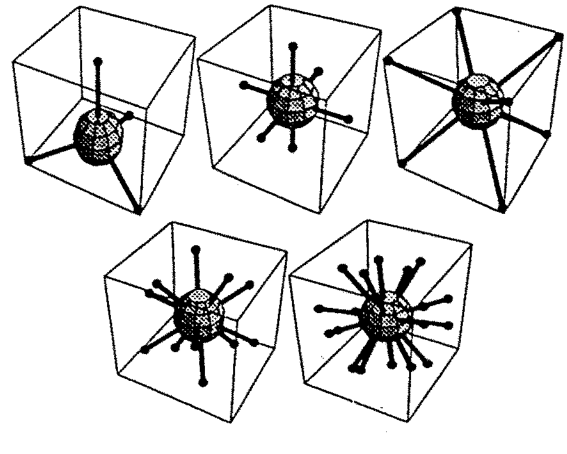

Fig. 8. Regular-polyhedron-type configurations.

$$\boldsymbol{K}''_G = \frac{\boldsymbol{\Omega}(\boldsymbol{\mu}_c)\boldsymbol{J}_c - \boldsymbol{J}_c\boldsymbol{\Omega}(\boldsymbol{\mu}_c)}{2}. \tag{24}$$

For the regular polyhedrons $\boldsymbol{J}_c$ is proportional to the unit tensor (Coxeter, 1973), and this eliminates $\boldsymbol{K}''_G$ from the analysis. As can be shown, the eigenvalues of the remaining matrix $\boldsymbol{K}'_G$ are defined as follows: $\lambda_1 = \boldsymbol{\rho}_c^{\mathrm{T}}\boldsymbol{f}_c$ and

$$\lambda_{2,3} = \frac{\boldsymbol{\rho}_c^{\mathrm{T}}\boldsymbol{f}_c \pm \sqrt{(\boldsymbol{\rho}_c^{\mathrm{T}}\boldsymbol{\rho}_c)(\boldsymbol{f}_c^{\mathrm{T}}\boldsymbol{f}_c)}}{2}. \tag{25}$$

By the Cauchy inequality

$$(\boldsymbol{\rho}_c^{\mathrm{T}}\boldsymbol{f}_c)^2 \le (\boldsymbol{\rho}_c^{\mathrm{T}}\boldsymbol{\rho}_c)(\boldsymbol{f}_c^{\mathrm{T}}\boldsymbol{f}_c), \tag{26}$$

and one of the eigenvalues will be always negative unless $\boldsymbol{\rho}_c = k\boldsymbol{f}_c$, which reduces the analysis to the case considered before.

4.2. *Internal-Force-Induced Stiffness*

Let us now consider stability due to the internal forces. To facilitate mathematical description of the forthcoming analysis and to cover the general case of n contact points, we will use another description for the elements α_{ij} of the vector α. Namely, it will be assumed that they are somehow ordered and can be addressed by only one subscript. The same rule will be kept for the correspondent vectors $\boldsymbol{r}_{ij}$ and $\boldsymbol{f}_{ij}$. The use of single indexed variables will be marked by bar sign, i.e., $\bar{\alpha}_l$ will correspond to some element α_{ij}.

As before, we start the analysis with considering the planar case. First, by direct summing up all α_{ij} as given by (14) and comparing the result with (6), we arrive at the following remarkable formula

$$K_I = \frac{1}{n}\sum_{i=1}^{N}\bar{\alpha}_i > 0, \tag{27}$$

where $N = n(n-1)/2$ is dimension of the vector α. In the planar case this condition is necessary and sufficient for stability under the internal forces.

Next, coming to the spatial case, we represent the internal forces as $\boldsymbol{f}_I = \boldsymbol{A}_o^{+}\alpha$ and substitute them into (5). After simplification we obtain

$$K_I = \sum_{i=1}^{N} \bar{\beta}_i\{(\bar{r}_i^T \bar{r}_i)I - \bar{r}_i \bar{r}_i^T\}, \tag{28}$$

where elements of the vector $\beta = \{\bar{\beta}_1, \ldots, \bar{\beta}_N\}^T$ are related to the components of the vector α by

$$\beta = (A_o A_o^T)^{-1}\alpha. \tag{29}$$

As can be easily seen, K_I has the structure of the inertia tensor of a system of points built on the vectors $\bar{r}_i$, with $\bar{\beta}_i$ playing the role of masses. Hence, N linear with respect to $\bar{\alpha}_i$ conditions of $\bar{\beta}_i \geq 0$ would be sufficient to guarantee that the matrix K_I is not negative definite. As will be shown below, they can be reduced to just two conditions—one linear and the other one quadratic—imposed on the elements of α.

At first, we will show that $\lambda_1 = \frac{1}{n}\sum_{i=1}^{N} \bar{\alpha}_i$ is valid eigenvalue of the matrix K_I. Indeed, computing the determinant of the matrix $K_I - \lambda_1 I$, we get $\det(K_I - \lambda_1 I) = \det(\sum_{i=1}^{N} \bar{\beta}_i \bar{r}_i \bar{r}_i^T)$. By the generalized Lagrange identity (Bellman, 1960) we have

$$\det(\sum_{i=1}^{N} \bar{\beta}_i \bar{r}_i \bar{r}_i^T) = \frac{1}{6} \sum_{i,j,k=1}^{N} \bar{\beta}_i \bar{\beta}_j \bar{\beta}_k (\det[\bar{r}_i \vdots \bar{r}_j \vdots \bar{r}_k])^2.$$

However, by construction of the virtual spring model, every three vectors $\bar{r}_i, \bar{r}_j, \bar{r}_k$ there are linearly dependent. Hence, $\det(K_I - \lambda_1 I) = 0$ and λ_1 is the eigenvalue of K_I. Thus, the condition (27) is also necessary for stability in the spatial case. As to the other two eigenvalues of K_I, we note that $\mathrm{tr}K_I = 2\lambda_1$, and therefore

$$\lambda_{2,3} = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - 4\gamma(\alpha)}}{2}, \tag{30}$$

where $\gamma(\alpha) = \det K_I / \lambda_1$ is the quadratic form of α. The stability is guaranteed if $\gamma = \alpha^T \Gamma_\alpha \alpha \geq 0$, provided that (27) holds true. Now, the remaining part of the analysis is to define the matrix Γ_α. Note here that γ can be also represented in terms of variables β, i.e., as $\gamma = \beta^T \Gamma_\beta \beta$. We can define elements of Γ_β with the use of the unit coefficients method, i.e., by computing $\det K_I$ and λ_1 with $\beta = \{0, \ldots, 0, 1, 0, \ldots, 0\}^T$ and $\beta = \{0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0\}^T$. Skipping all the intermediate results, we can show that

$$\{\Gamma_\beta\}_{ij} = \begin{cases} 0 & \text{if } i = j \\ (\bar{r}_i \times \bar{r}_j)^T (\bar{r}_i \times \bar{r}_j) & \text{otherwise} \end{cases} \tag{31}$$

It is interesting that the geometric meaning of the off-diagonal elements of Γ_β is that $\{\Gamma_\beta\}_{ij} = S_{ij}^2/4$, where S_{ij} is the area of the triangle built on the vectors $\bar{r}_i$ and $\bar{r}_j$. Note that Γ_β is singular sign-indefinite form. We can show that in the space of variable β this quadric is represented by a cone.

Now, having defined Γ_β we can return to Γ_α. Taking into account the relation between α and β, we obtain

$$\Gamma_\alpha = (A_o A_o^T)^{-1} \Gamma_\beta (A_o A_o^T)^{-1} \tag{32}$$

Finally, recalling the relation between α and f_I, we can formulate the second stability condition in terms of the internal forces f_I. It reads

$$f_I^T A_o^+ \Gamma_\beta (A_o^+)^T f_I \geq 0. \tag{33}$$

This completes our analysis. Note that geometrically, regardless of whether we conduct the stability analysis in terms of α, β, or f_I, the stability area is defined by intersection of the plane (27) with the cone (33).

5. CONCLUSION

The problem of the rotational grasping stability under the constant force loading has been addressed in this paper. The structure of the stiffness tensor has been represented through the contact force decomposition. Two conditions of stable grasping under the internal force loading has been obtained in the analytical form. However, only one condition of stability, formulated in terms of the geometric and gravity centers, has been established for the gravity-induced stiffness.

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